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TOPOLOGY AND EINSTEIN KAEHLER METRICS

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Introduction

The main result of this paper is Theorem 4.1 which gives some interesting topological restrictions for the construction of Einstein Kaehler metrics on complex 2-manifolds. Theorem 4.1 follows quite easily from the classical index theorems and recent invariance theory calculations.

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1. Invariance theory

Let P be a map from Riemannian manifolds M to complex valued functions on M. We say that P is an invariant polynomial of order k in the derivatives of the metric g if for any $m \in M$ and smooth coordinate system normalized at origin m, $g_{ij}(m) = \delta_{ij}$, P may be expressed as a polynomial in the derivatives of g_{ij} with respect to the $\partial/\partial x_s$ such that any monomial of P contains precisely k derivatives.

These invariants have been studied extensively. However for this paper we will need only the following elementary proposition:

Proposition 1.1. *Let (M, g) be a Riemannian manifold of dimension greater than or equal to four. Denote R its curvature tensor, ρ its Ricci tensor, τ its scalar curvature, and Δ its Laplace oparator. Then $\|R\|^2, \|\rho\|^2, \tau^2, \Delta\tau$ form a basis for the invariants of order four in the derivatives of g.*

Proof. See for example [2, p. 77].

At a point m and normalized coordinate system centered at m we have $g_{ij}(m) = \delta_{ij}$ and therefore

$$\|R\|^2 = \sum_{i,j,k,l} (R_{ijkl})^2 , \qquad \|\rho\|^2 = \sum_{j,k} \left(\sum_i R_{ijik} \right)^2 ,$$
$$\tau^2 = \left(\sum_{i,j} R_{ijij} \right)^2 , \qquad \Delta\tau = \sum_{i,j,k} R_{ijij,kk} .$$

Now let Q be a map from hermitian manifolds N to complex valued func-

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tions on N. We say that Q is an invariant polynomial of order k in the derivatives of the hermitian metric h if for any $n \in N$ and holomorphic coordinate system normalized at origin $n, h_{\alpha\bar\beta}(n) = \delta_{\alpha\beta}$, Q may be expressed as a polynomial in the derivatives of $h_{\alpha\bar\beta}$ with respect to $\partial/\partial z_\gamma, \partial/\partial \bar z_\gamma$ such that any monomial of P contains precisely k derivatives. Clearly the invariant polynomials in the derivatives of h contain as a subset the invariant polynomials in the derivatives of the underlying Riemannian metric g.

For the invariants of order four on a Kaehler manifold the following result holds:

Proposition 1.2. *Let N be a Kaehler manifold of complex dimension greater than or equal to two. Denote R the curvature tensor of the underlying Riemannian metric g. Then the invariants $\|R\|^2, \|\rho\|^2, \tau^2, \Delta\tau$ of Proposition 1.1 form a basis for the invariants of order four in the derivatives of the hermitian metric h.*

Proof. Gilkey and Sacks [5] showed that the invariants of order four span a vector space of dimension four and gave an explicit basis. The author [4] then observed that the Riemannian invariants are linearly independent on Kaehler manifolds. Thus $\|R\|^2, \|\rho\|^2, \tau^2, \Delta\tau$ form a basis for the hermitian invariants on a Kaehler manifold of complex dimension greater than or equal to two.

For a more complete account of the invariance theory of hermitian manifolds the reader may consult [3].

2. The Riemannian case

Let (M, g) be a four-dimensional Riemannian manifold. Then as is well known

$$\chi(M) = \int_M *\chi(\Omega) ,$$

where $\chi(M)$ denotes the Euler characteristic of M, and $*\chi(\Omega)$ is the Gauss-Bonnet integrand.

In particular, $*\chi(\Omega)$ is an invariant polynomial of order four in the derivatives of the metric and is thus a linear combination of the invariants of Proposition 1.1. Since no derivatives of the curvature tensor appear, $*\chi(\Omega)$ is a linear combination of $\|R\|^2, \|\rho\|^2, \tau^2$ only. The appropriate coefficients may be deduced by computation on specific manifolds. For this purpose the reader should consult the appropriate entries in the table of § 4. We have then

$$\chi(M) = \frac{1}{32\pi^2}\left(\int\|R\|^2 - 4\int\|\rho\|^2 + \int\tau^2\right) .$$

Now suppose M is Einsteinian. Recall that an n-dimensional Riemannian manifold is said to be Einsteinian if M has constant scalar curvature τ and $\rho = (\tau/n)g$. Then $\|\rho\|^2 = \tau^2/n$. Since M is four dimensional, $\tau^2 = 4\|\rho\|^2$ and

$$\chi(M) = \frac{1}{32\pi^2}\int \|R\|^2 .$$

This yields the following theorem of Berger [1]:

Theorem 2.1. *Let M be a four-dimensional differentiable manifold. If M admits an Einstein metric, then $\chi(M) \geq 0$. Further, if $\chi(M) = 0$, then any Einstein metric on M is flat.*

Remark 1. Our proof is somewhat different from that of Berger and apparently more simple minded.

Remark 2. Hitchin [6] showed that a necessary condition for a four-dimensional orientable manifold to admit an Einstein metric is $\chi(M) \geq \frac{3}{2}|\text{sign}\,(M)|$ where sign (M) denotes the signature of M. This strengthens Theorem 2.1 in the orientable case.

3. An inequality

For our work in § 4 we will need

Proposition 3.1. *Let M be a complex Kaehler manifold of real dimension n. Denote R its Riemann curvature tensor and τ its scalar curvature. Then*

$$\|R\|^2 \geq \frac{8}{n(n+2)}\tau^2 .$$

If τ is constant and equality holds, then M has constant holomorphic sectional curvature $4\tau/[n(n+2)]$. Finally if $n > 2$, then equality implies that τ is constant.

Proof. Let ω be the tensor defined by $\omega(X, Y) = g(X, JY)$ where g is the Riemannian metric and J the almost complex structure on TM. Denote T the tensor whose components in the real coordinate system associated with any local holomorphic coordinate system are

$$T_{ijkl} = R_{ijkl} - \frac{\tau}{n(n+2)}[(g_{ik}g_{jl} - g_{il}g_{jk}) + (\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk} + 2\omega_{ij}\omega_{kl})] ,$$

$$0 \leq \|T\|^2 = \|R\|^2 - \frac{8}{n(n+2)}\tau^2 .$$

This proves the first part of the proposition.

If equality holds then $T = 0$. So when $\partial/\partial x_j = J(\partial/\partial x_i)$,

$$0 = T_{ijij} = R_{ijij} - \frac{4\tau}{n(n+2)}$$

for any normalized coordinate system $g_{ij} = \delta_{ij}$. Then

$$R_{ijij} = \frac{4\tau}{n(n+2)} ,$$

and by considering all possible normalized coordinate systems we conclude that M has constant holomorphic sectional curvature $4\tau/[n(n+2)]$ when τ is constant.

Now assume $n > 2$ and equality holds. As above $T = 0$ so

$$R_{ijkl} = \frac{\tau}{n(n+2)}[(g_{ik}g_{jl} - g_{il}g_{jk}) + (\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk} + 2\omega_{ij}\omega_{kl})] .$$

Consider the second Bianchi identity $R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} = 0$ for k, l, m distinct. Substitution and contraction of the pairs (j, m) and (i, k) yield $\tau_{,l}(-1 - 3\omega_{km}^2) = 0$ in any normalized holomorphic coordinate system $g_{ij} = \delta_{ij}$. Thus $\tau_{,l} = 0$ and $d\tau = 0$, so τ is a constant.

4. The Kaehler case

Let (M, h) be a complex Kaehler manihold of complex dimension two. Denote $a(M)$ the arithmetic genus of M, and sign (M) the signature of M with orientation induced by the almost complex structure. Then

$$a(M) = \frac{1}{12}\int *(c_1^2 + c_2) , \qquad \text{sign}\,(M) = \frac{1}{3}\int *p_1 ,$$

where c_1, c_2 are the Chern forms of M, p_1 its first Pontriagin form, and $*$ the Hodge star operator associated with the Hermitian structure.

In particular, each of the integrands is an invariant polynomial of order four in the derivatives of h. Thus these integrands are linear combinations of the invariants from Proposition 1.2 and linear combinations of τ^2, $\|\rho\|^2$, and $\|R\|^2$ only since no derivatives of the curvature tensor appear. The appropriate coefficients may be deduced by computation on specific manifolds. For this purpose we present a table below which the reader may verify at his convenience.

	$S^2 \times S^2$	$S^2 \times T^2$	CP^2
$a(M)$	1	0	1
$\chi(M)$	4	0	3
sign (M)	0	0	1
$\|R\|^2$	8	4	192
$\|\rho\|^2$	4	2	144
τ^2	16	4	576
vol (M)	$16\pi^2$	4π	$\pi^2/2$

Here T^2 is normalized to have unit volume, and S^2, CP^2 are given their usual metrics of constant sectional curvature one and constant holomorphic sectional curvature four respectively. Then we deduce

$$a(M) = \frac{1}{12(32)\pi^2}\left(\int \|R\|^2 - 8\int \|\rho\|^2 + 3\int \tau^2\right),$$

$$\operatorname{sign}(M) = \frac{-1}{48\pi^2}\left(\int \|R\|^2 - 2\int \|\rho\|^2\right),$$

and recall from § 2:

$$\chi(M) = \frac{1}{32\pi^2}\left(\int \|R\|^2 - 4\int \|\rho\|^2 + \int \tau^2\right).$$

From these formulas one sees that $\chi(M) + \operatorname{sign}(M) = 4a(M)$ which also follows from the definitions of $\chi(M)$, $a(M)$ and the Hodge signature theorem.

If M is Einstein Kaehler, then $\tau^2 = 4|\rho|^2$ and our formulas reduce to

$$\begin{aligned} a(M) &= \frac{1}{12(32)\pi^2}\left(\int \|R\|^2 + 4\int \|\rho\|^2\right), \\ \operatorname{sign}(M) &= \frac{-1}{48\pi^2}\left(\int \|R\|^2 - 2\int \|\rho\|^2\right), \\ \chi(M) &= \frac{1}{32\pi^2}\int \|R\|^2 . \end{aligned} \tag{4.1}$$

This yields the following

Theorem 4.1. *Let M be a complex analytic manifold of complex dimension two. If M admits an Einstein Kaehler metric, then we have:*

(a) $0 \le 3a(M) \le \chi(M) \le 12a(M)$. *If $a(M) = 0$, then all Einstein Kaehler metrics are flat. If $\chi(M) = 3a(M)$, then all Einstein Kaehler metrics have constant holomorphic sectional curvature. If $\chi(M) = 12a(M)$, then all Einstein Kaehler metrics are Ricci flat.*

(b) $-2\chi(M) \le 3\operatorname{sign}(M) \le \chi(M)$. *If $3\operatorname{sign}(M) + 2\chi(M) = 0$, then all Einstein Kaehler metrics are Ricci flat. If $\chi(M) = 3\operatorname{sign}(M)$, then all Einstein Kaehler metrics have constant holomorphic sectional curvature.*

(c) $-8a(M) \le \operatorname{sign}(M) \le a(M)$. *If $\operatorname{sign}(M) + 8a(M) = 0$, then all Einstein Kaehler metrics are Ricci flat. If $\operatorname{sign}(M) = a(M)$, then all Einstein Kaehler metrics have constant holomorphic sectional curvature.*

Proof. By Proposition 3.1, $\|R\|^2 \ge \frac{1}{3}\tau^2 = \frac{4}{3}\|\rho\|^2$ with equality only for metrics of constant holomorphic sectional curvature. Theorem 4.1 then follows directly from formulas (4.1).

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